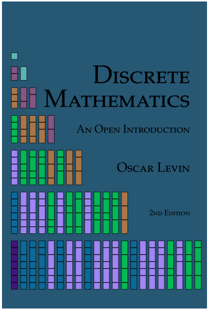
Shifts in Perspectives for Counting

Oscar Levin

University of Northern Colorado

Discrete Mathematics in the Undergraduate Curriculum
Joint Mathematics Meeting
January 10, 2018

Discrete Mathematics: an Open Introduction



A free, open source introductory discrete textbook with interactive online, pdf ebook, and print editions.

Goal: improve exposition for upcoming 3rd edition. In particular *counting*!

An Example

How many lattice paths are there from $(0, 0)$ to $(8, 5)$ that pass through $(5, 2)$?

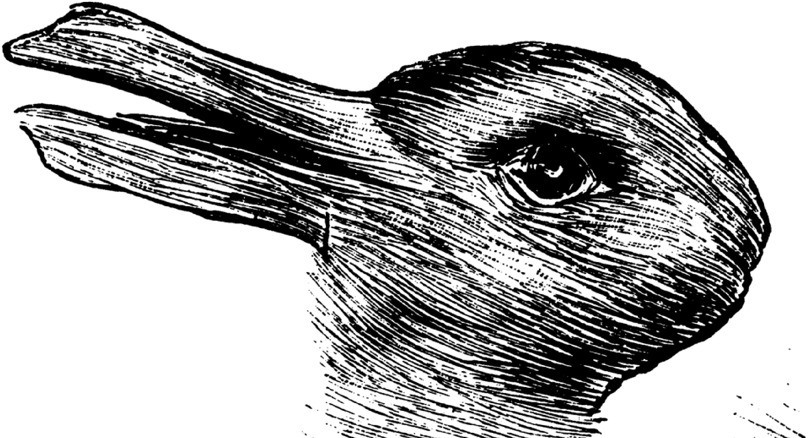
Number of paths from $(0, 0)$ to $(5, 2)$: $\binom{7}{5}$

Number of paths from $(5, 2)$ to $(8, 5)$: $\binom{6}{3}$

Final answer: $\binom{7}{5} + \binom{6}{3}$???

Why is counting so hard for students

- No way to check solutions
- Multiple ways to write solutions
- Every problem is a word problem
- Differences in problem types are subtle
- Instructors are better mathematicians than their students



Counting *is* hard!

In Yatzee you roll 5 identical 6-sided dice. How many different rolls are possible?

Hint: Counting compositions (natural number solutions to $x_1 + x_2 + \cdots + x_n = k$) uses “stars and bars.”

Hint: The Yatzee question is like counting multisets.

Hint: Counting multisets is just like counting compositions?

Multisets vs Compositions

How many multisets of size 5 are there from $\{1, 2, \ldots, 8\}$?

or

How many solutions are there to

$x_1 + x_2 + \cdots + x_n = k x_1 + x_2 + \cdots + x_8 = 5$?

$$\{3, 3, 5, 6, 8\}$$

$$0 + 0 + 2 + 0 + 1 + 1 + 0 + 1 = 5$$

$$|| * * || * | * ||*$$

Total: $\binom{5+7}{5}$

Like Giving Candy to Babies

How many ways can you distribute 5 identical candies to 8 kids?

$$\binom{5+7}{5}$$

How many ways can you distribute 5 *distinct* candies to 8 kids?

$$8^5$$

How many ways can you distribute 5 identical candies to 8 kids so that no kid gets more than one candy?

$$\binom{8}{5}$$

Subsets and Bitstrings

How many sequences from $\{a, b, c, \ldots, h\}$ have length 5?

How many subsets of $\{a, b, c, \ldots, h\}$ have size 5?

Embrace It: Combinatorial Proofs

Prove that $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$.

Meta-Ambiguities

The previous examples illustrate (one reason) why counting is conceptually challenging. But why do students make simple mistakes?

Why add instead of multiply?

Why apply permutations when repeats are allowed?
(No, that is NOT the distinction between combinations and permutations.)

There are two interpretations of the sum and product principle:

1. If task A can be completed in m ways and task B can be completed in n ways then....
2. If A and B are disjoint sets, then $|A \cup B| = |A| + |B|$ and $|A \times B| = |A| \cdot |B|$.

- It is useful to think of describing an event in steps.
- It is important to think of the set of outcomes that we are counting.

The Sum of all Paths

How do you get a path from $(0, 0)$ to $(8, 5)$ that passes through $(5, 2)$?

You must take a path from $(0, 0)$ to $(5, 2)$ (can happen in $\binom{7}{5}$ ways) and **ADD** on a path from $(5, 2)$ to $(8, 5)$ (can happen in $\binom{6}{3}$ ways).

Combinations and Permutations and Repeats

How does the set of permutations relate to the set of combinations?

Consider the list of all permutations. Compare to combinations. Notice that a bunch of permutations all count as the same permutation. So when we count the combinations, we want to get rid of these **REPEATED** outcomes.

It gets worse!

How many license plates consist of 3 letters followed by 3 numerical digits?

$$\_\_\_ \; \_\_\_ \; \_\_\_ \quad \_\_\_ \; \_\_\_ \; \_\_\_$$

$$26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10$$

Another...

Why is $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$?

Think bit strings. $\binom{n}{k}$ is the number of n-bit strings of weight k (call this set $\mathbb{B}_k^n$). Each string either starts with a 0 or a 1. We might write:

$$\mathbb{B}_k^n = 1\mathbb{B}_{k-1}^{n-1} \cup 0\mathbb{B}_k^{n-1}$$

Then

$$|\mathbb{B}_k^n| = |\{1\}| \cdot |\mathbb{B}_{k-1}^{n-1}| + |\{0\}| \cdot |\mathbb{B}_k^{n-1}|.$$

But we have $|\mathbb{B}_k^n| = \binom{n}{k}$.

Meta-Ambiguities

- It is easy to conflate individual outcomes with the set of outcomes.
  (Especially when giving examples of outcomes.)
- It is easy to conflate the set of outcomes with the number of outcomes.
  (Especially when we count events using tasks.)

But what can you do?

Nothing!

Let counting be hard!

. . . but help students understand *why*.

Combinatorics as a bridge

- A matrix represents a system of equations and a linear transformation.
- A group table tells you how to multiply and describes a permutation.
- A sequence of functions can converge pointwise and uniformly.

The End

Thanks!

Slides and open textbook:

discrete.openmathbooks.org/talks.php