Using Proofs to Introduce Logic

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Background

Discrete Mathematics (Math 228) at UNC is (essentially) our bridge course.

Students: mostly pre-service math teachers.

Topics: Counting, Sequences, Graph Theory, Logic & Proofs.

Logic & Proofs

We spent 2 weeks on the subject in the middle of the semester.

Goals for logic:

- Survey logic as a topic in mathematics.
- Truth-tables, logical equivalence, deduction rules.
- Predicate logic and quantifiers.
- Converse, contrapositive, negation.

Goals for proofs:

- Introduce styles of proof and conventions.
- Direct proof, proof by contrapositive, proof by contradiction.
- Induction & combinatorial proofs (done earlier).
- Improve mathematical writing.

Personal goal:

- Students will see the connection between logic and proofs.

(Almost) Introductory Activity

Students are given four "proofs" for the statement,

> *If ab is even, then a is even or b is even.*

Students decide (in groups) which of the proofs are valid.

Give it a try!

1 and 2

1. Suppose $a = 2k + 1$ (a is odd) and $b = 2m + 1$ (b is odd). Then

$$\begin{aligned} ab &= (2k+1)(2m+1) \\ &= 4km + 2k + 2m + 1 \\ &= 2(2km + k + m) + 1 \end{aligned}$$

   which proves that ab is odd if a and b are odd. Therefore, if ab is even, then a or b is be even.

2. Assume that a or b is even - say it is a. That is, $a = 2k$ for some integer k. Then

$$\begin{aligned} ab &= (2k)b \\ &= 2(kb) \end{aligned}$$

   which means that ab is even. The case where b is even is identical. Therefore, if ab is even then a is even or b is even.

Students consensus: 1 is invalid, 2 is valid.

Logic to the rescue

Translate: "If ab is even, then a is even or b is even"

$$P \to (Q \vee R)$$

$$P \to S$$

In proof 1: we assume $\neg Q \wedge \neg R$ (i.e., $\neg S$) and deduce $\neg P$.

$$\therefore (\neg Q \wedge \neg R) \neg S \to \neg P$$

In proof 2: we assume $Q \vee R$ and deduce P.

$$\therefore S \to P$$

By contradiction

3. Suppose that ab is even but a and b are both odd. Namely, $ab = 2n$, $a = 2k + 1$ and $b = 2j + 1$ for some integers n, k, and j. Then

$$\begin{aligned} 2n &= (2k+1)(2j+1) \\ 2n &= 4kj + 2k + 2j + 1 \\ n &= 2kj + k + j + \frac{1}{2} \end{aligned}$$

   But since $2kj + k + j$ is an integer, this says that the integer n is equal to a non-integer, which is impossible. Therefore, if ab is even then a or b must be even.

Note: what is the assumption?

$$P \wedge (\neg Q \wedge \neg R) \equiv \neg(P \rightarrow (Q \vee R))$$

Another one

4. Let ab be an even number, say $ab = 2n$, and a be an odd number, say $a = 2k + 1$. Then

$$
\begin{aligned}
ab &= (2k + 1)b \\
2n &= 2kb + b \\
2n - 2kb &= b \\
2(n - kb) &= b
\end{aligned}
$$

Therefore b must be even.

$$P \to (Q \vee R) \equiv (P \wedge \neg Q) \to R$$

The good

- Naturally motivated logic.
- Students took ownership.
- Illustrated connection between logic and proofs.
- Highlighted standard proof formats from the right direction.

The bad

- Long activity.
- Confusion over $Q \vee R$ vs S.
- Students hung up on algebra and meaning of even/odd.
- Statements about parity are trivial and boring.

Some fixes

If ab is even, then a is even~~is even~~ or b is even.

Clean up proofs to make assumptions and conclusions obvious.

Think up more interesting statement (under construction)

Other ideas?

Thanks!

Slides and handout:

discretetext.oscarlevin.com/talks.php