Knights and Knaves in the Classroom

Oscar Levin

University of Northern Colorado

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A Classic

While walking through a fictional forest, you encounter three trolls guarding a bridge. Each is either a knight, who always tells the truth, or a knave, who always lies. The trolls will not let you pass until you correctly identify each as either a knight or a knave. Each troll makes a single statement:

> Troll 1: Only one of us is a knave.
>
> Troll 2: No, only one of us is a knight.
>
> Troll 3: We are all knaves.

Which troll is which?

Motivation

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- Explanations are proofs.
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- Promotes logical reasoning and careful thinking.
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- Illustrates proof by contradiction.
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- Exposes students to real world applications of modern mathematics.

Except. . .

- Where's the logic?

Connectives

You stumble upon two trolls playing Stratego®. They tell you:

Troll 1: If we are cousins, then we are both knaves.

Troll 2: We are cousins or we are both knaves.

Could both trolls be knights?

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Hint 2: If Troll 1 was a knight, what could you conclude?

Table Time

Troll 1: If we are cousins, then we are both knaves.

Troll 2: We are cousins or we are both knaves.

Could both trolls be knights?

P: We are cousins. Q: We are both knaves.

P	Q	$P \to Q$	$P \vee Q$
T	T	T	T
T	F	F	T
F	T	T	T
F	F	T	F

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Quantified Trolls

You come across four trolls playing bridge. They declare:

Troll 1: All trolls here see at least one knave.
Troll 2: I see at least one troll that sees only knaves.
Troll 3: Some trolls are scared of goats.
Troll 4: All trolls are scared of goats.

Are there any trolls that are not scared of goats?

Quantified Trolls Solution

Troll 2: I see at least one troll that sees only knaves.

This cannot be true. If it was, then everyone this supposed troll sees would be a knave, including Troll 2. So Troll 2 is a knave and his statement is false. Everyone he sees must see at least one knight. In particular, there must be at least two knights in the group.

Troll 1: All trolls here see at least one knave.

This cannot be false. If it was, then someone would see only knights. But in this case, both Troll 2 and Troll 1 would be knaves. Thus there are at least two knaves. There are two knights and two knaves.

Troll 3: Some trolls are scared of goats; Troll 4: All trolls are scared of goats. If Troll 4 is a knight, so is Troll 3.

Thus some trolls are scared of goats but not all trolls, so some are not scared of goats.

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A better classic

Troll 1: If I am a knave then there are exactly two knights here.

Troll 2: Troll 1 is lying.

Troll 3: Either we are all knaves or at least one of us is a knight.

Which troll is which?

Induction

You find yourself face-to-face with 13 trolls.

Luckily for you, each troll makes a statement. The trolls make almost identical statements:

> *By the time I'm finished speaking, you will have heard x of us lie to you*

where x is some number. However, no two trolls use the same number for x. Additionally, the first 12 trolls pick numbers for x such that it is impossible to deduce whether the troll who just spoke is a knight or a knave. After hearing the statement of the 13th troll, you can deduce the status of all 13 trolls.

Which trolls are knights and which are knaves?

?

Thanks

Slides and Problems:

`discretetext.oscarlevin.com/talks.php`

Induction solution

Troll 12 (the second to last to speak) is a knight, all the other trolls are knaves. The first troll uses $x = 0$, the second troll uses $x = 1$, and so on, except for the last troll who uses $x = 13$.

Troll 1 cannot use $x = 1$, because if he is a knight, that would make him a liar, and if he is a knave, that would make his statement true.

Troll 2 can only use $x = 1$ or 2 (zero has been used and any other number would identify him immediately has a knave). However, if he took $x = 2$, then if he were a knight, he would be lying, so he must be a knave. That would be consistent – it would be false that there were two liars and we could deduce the class of both trolls, contrary to stipulation.

Induction solutions (cont)

Similar reasoning (and induction) show that the only choice for troll number n is for $x = n - 1$, lest we be able to deduce the status of all the trolls.

Since we can make this deduction at troll 13, he must use $x = 13$. This will be false, so there are fewer than 13 knaves. If there are fewer than 12 knaves though, two of trolls 1-12 would be knights. However each of their statements imply that all the previous trolls are knaves, so one of them must be lying. So there is only one knight. If that one knight was not troll number 12, then troll 12's statement would be true, a contradiction.

back to puzzle